Intercept Probability of Satellites in Circular Orbits

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An analytical and computational approach is provided to calculate the probability that a designated transmitter satellite in circular orbit will intercept a particular receiver satellite also in circular orbit. The term intercept implies that the transmitter satellite passes within some minimum distance of the receiver satellite for a minimum period of time. The probability calculated is the probability of interception within some maximum time interval. This paper discusses the dynamics and definitions leading to intercept probability calculations and compares the predicted intercept probabilities to estimates determined through Monte Carlo techniques.

I. Introduction

THE problem posed in this paper can be defined as a form of communications problem. Assume that one satellite in a circular orbit must transmit information to another satellite in circular orbit within some specified period of time, T_I. Furthermore, assume that a transmission time of at least t_t s must be allowed, and the transmitter must be within a range R of the receiver for the communication to take place. Then, an important parameter of this communications problem is the probability that the transmitter will be within R for at least t_t within T_I starting from some random initial time. Throughout the remainder of this paper the transmitter will be called a platform and the receiver will be called the target. An intercept will be defined to be the event of the platform passing within the distance R of the target for at least t_t s. Therefore, the problem can be restated to be the calculation of the probability of interception within the interval T_I assuming a random starting time. An obvious solution to this problem is to simply simulate the trajectories of the platform and target and, using random initial positions of the satellites, estimate the probability of intercept using a Monte Carlo technique. However, Monte Carlo techniques, although effective, provide little insight into the understanding of the problem.

A similar problem involving the collision of satellites has been studied by several authors.[1-3] A simplified collision problem treats essentially the same types of parameters: the probability that two orbiting objects will pass within some specified distance within some specified time. However, the distance is typically on the order of meters in the collision problem and kilometers in the intercept problem, and the time is on the order of months and years in the collision problem and hours in the intercept problem. Perhaps a more significant difference is that an intercept is an event that occurs over a time interval, whereas a collision is essentially a point event. Despite these differences both problems share many areas of common interest.

The primary purpose of this paper is to provide an analytical model for understanding and calculating the probability of intercept. It will be assumed throughout this paper that the platform and target are both in circular orbits. Although this assumption appears rather restrictive, it does lead to a tractable problem and provides a close approximation for low eccentricity orbits. Furthermore, the solution also provides some interesting and potentially valuable insights into the variation of intercept probability. It should be noted at this point that the probability calculations also ignore the effects of precession. The error introduced by this omission will be small if the time T_I is small relative to the period of the difference of the platform and target precession frequencies.

The analysis which precedes the intercept probability calculations is divided into three parts. The first part (Sec. II) will summarize the simple dynamics of the platform and target for the purpose of defining useful terms. Section II also will define two key variables and their relationship to the orbital parameters, target range, T_I, and t_t. These variables form the basis of the probability model. The second part (Sec. III) will develop the probability model. Finally, the intercept probability calculations will be made, and the implications of the resulting curves will be discussed.

II. Dynamics and Key Variables

This section develops the simple equations which determine the relative motion of two satellites, a platform and a target, which are in circular orbits and do not necessarily lie in the same plane. This development serves three purposes. It provides 1) an opportunity to define the variables and terms used in this paper, 2) a simple method for introducing two key parameters in the analysis, and 3) a foundation for the probability discussions in Sec. III.

The position of any satellite relative to an Earth-centered inertial coordinate system which has its origin at the center of the Earth and has its x and y axes in the equatorial plane of the Earth can be defined by a frequently used six orbital element set (a, e, i, W, g, v) (Fig. 1) and an epoch. Here a is the semimajor axis of the orbit, e the eccentricity of the orbit, i the angle of inclination, W the right ascension of the ascending node, g the argument of perigee, v the true anomaly angle, and T the epoch when the satellite is at anomaly position, v. Since the problem being discussed assumes circular orbits and a constant right ascension, the orbital element sets can be simplified considerably.

The only important variables are the platform orbital radius R_P, the target orbital radius R_T, the angle between the orbital planes (i_R = relative angle of inclination), and the anomaly angles for each satellite (v_P = platform anomaly angle and v_T = target anomaly angle). The remaining orbital elements, W and i, define the orientations of the orbital planes with respect to the equatorial plane and each other. Since circular orbits have no unique line of symmetry, the line of the two orbital planes can be chosen at any convenient location. Setting $W=0$ for both orbits, the line of intersection will coincide with the line of nodes of both orbits. Also, since only the relative angle of inclination is important in the calculations, the inclination angle of one of the orbits can be chosen arbitrarily. Therefore, to simplify the algebra, arbitrarily set i_P (the platform's inclination angle) to zero which implies that i_T (the target's inclination angle) is equal to i_R. The resulting orbital element set for the platform is $(R_P, 0, 0, 0, 0, v_P)$, and the orbital element set for the target is $(R_T, 0, i_R, 0, 0, v_T)$.

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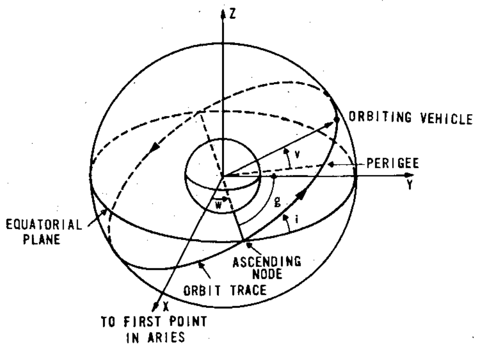

Fig. 1 Orbital elements.

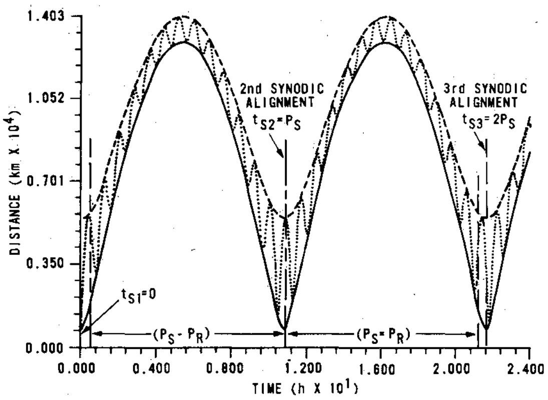

Fig. 2 Regional minimum.

Then the distance r between the platform and target is defined by

$$r^2 = R_P^2 + R_T^2 - R_P R_T\big((1+\cos i_R)\cos((w_P - w_T)t + v_0) + (1-\cos i_R)\cos((w_P + w_T)t + v_D)\big) \tag{1}$$

where

$$v_0 = v_{P0} - v_{T0} \tag{2}$$

$$v_D = v_{P0} + v_{T0} \tag{3}$$

and v_{P0} is the initial value of v_P, v_{T0} the initial value of v_T, w_P the angular rate of platform rotation, and w_T the angular rate of target rotation.

Equation (1) is plotted in Fig. 2 as the dotted curve. The curve is characterized by many local extrema which occur with a frequency $w_P + w_T$ and a period

$$P_R = \frac{P_P P_T}{P_P + P_T} \tag{4}$$

where P_P is the period of the platform orbit and P_T the period of the target orbit. The upper and lower envelopes of this curve vary with a frequency of $|w_P - w_T|$ and period

$$P_S = \frac{P_P P_T}{P_P - P_T} \tag{5}$$

where P_S is the synodic period. Let a synodic alignment be defined as the event when the position vectors of the target and platform have the same true anomaly angle (i.e., $v_P = v_T$). Then the time between successive synodic alignments is the synodic period, and an intercept will occur near a synodic alignment if, and only if, the synodic alignment is sufficiently

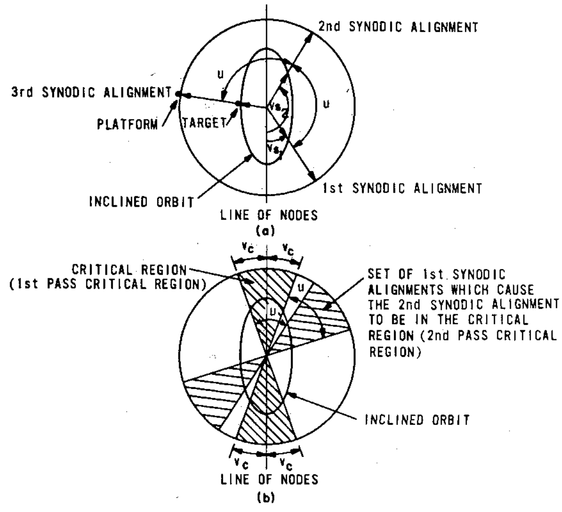

Fig. 3 Multiple-pass interception sets.

close to a node. The following discussion will define the set of synodic alignments that is sufficiently close to a node and is therefore preceded by, followed by, or coincident with, an intercept. Notice that an intercept is possible only if $|R_P - R_T| < R$. This will be assumed in the next discussion.

Let v_S be the anomaly angle that identifies the location of a synodic alignment relative to the line of nodes (i.e., $v_P = v_T = v_S$). Furthermore, let

$t_{\min}$ = the time the regional minimum is reached
t_1 = the time less than but nearest to $t_{\min}$ such that $r = R$ for a particular choice of R_P, R_T, i_R, and $v_P = v_T = v_S$
t_2 = the time greater than but nearest to $t_{\min}$ such that $r = R$ for a particular choice of R_P, R_T, i_R, and $v_P = v_T = v_S$

Then, for given R_P, R_T, R, and i_R, define the critical angle v_C to be the value of v_S (between 0 and 90 deg) for which $t_2 - t_1 = t_t$. Notice that if v_S is less than v_C, then $t_2 - t_1$ will be greater than t_t; and if v_S is greater than v_C, $t_2 - t_1$ will be less than t_t. Then, by symmetry, the set of synodic alignments that is in the vicinity of an intercept is

$$(v_S| - v_C < v_S < v_C \quad \text{or} \quad \pi - v_C < v_S < \pi + v_C) \tag{6}$$

This set will be called the "first-pass critical region" or simply "critical region" throughout the remainder of this paper. Any synodic alignment that occurs in the critical region corresponds to a nearby intercept (Fig. 3).

Given R_P, R_T, and i_R, the critical region is determined by finding the values of v_C, t_1, and t_2 (if they exist) which satisfy Eq. (1) with $v_P = v_T = v_C$ and $t = t_1$ (or t_2), and which also satisfy $t_2 - t_1 = t_t$. In general, the solution to the resulting three equations must be obtained numerically. However, an expression was obtained in Ref. 4 that is a reasonable approximation of v_C with rather modest constraints on i_R and $|R_P - R_T|$.

$$v_C = \frac{w_P + w_T}{|w_P - w_T|}\,\frac{d+f}{2f}\left[\frac{|w_P - w_T|}{2}t_t - \cos^{-1}\left\{\frac{R_P^2 + R_T^2 - R^2}{R_P R_T - (1+\cos i_R)} - \frac{1-\cos i_R}{1+\cos i_R}\cos\frac{(w_P + w_T)}{2}t_t\right\}\right] \tag{7}$$

where

$$d = (1 - \cos i_R)(w_P - w_T)^2 \tag{8}$$

and

$$f = (1 - \cos i_R)(w_P - w_T)^2 \tag{9}$$

For low Earth orbits, the constraints that give roughly 6% agreement with calculated values of v_C are $20 < i_R < 90$ and $|R_P - R_T| < 0.9R$. The primary purpose for presenting this approximation is to give some idea of the type of relationship that exists among the variables.

Once the critical angle is determined, the set of synodic alignments associated with an intercept is also determined. In other words, given one synodic alignment, an intercept has occurred or will occur on the first pass if the first synodic alignment is within the critical region.

The next logical step is to consider the possibility that T_I is large enough to allow two synodic alignments. In this case, there are three alternatives: 1) the first or second synodic alignment is in the critical region and an intercept occurs, 2) both the first and second synodic alignments are in the critical region and two intercepts occur, 3) neither the first nor second synodic alignment is in the critical region and no intercept occurs. Since this paper is concerned with intercepts, the intercept problem reduces to finding the set of first synodic alignments that results in an intercept on the first and/or second pass. The set of first synodic alignments that results in an intercept on the first pass is given by Eq. (6). All that remains is to find the set of first synodic alignments that results in an intercept on the second pass (equivalently, the second synodic alignment is in the critical region). This, in turn, depends on the angle between successive synodic alignments.

The synodic phase shift is the angle u between successive synodic alignments. Let v_{Si} be the true anomaly of the target and platform at the ith synodic alignment. Then, assuming $v_{P0} = v_{T0} = v_{SI}$ at $t = 0$, the anomaly for the target and platform are equal again after one synodic period and

$$u = v_{S2} - v_{SI} = w_T P_S - 2k_T\pi = w_P P_S - 2k_P\pi \tag{10}$$

where k_T and k_P are the smallest nonnegative integers for which u is less than 2π, and v_{S2} is the angle of second synodic alignment (Fig. 3). Note that in going from one synodic alignment to another, one of the satellites rotates through one full revolution more than the other. Therefore,

$$k_T + k_P = 2j + 1 \qquad i = 0, 1, 2, \ldots \tag{11}$$

and adding Eq. (10),

$$u = \frac{(w_P + w_T)}{2} P_S - (2j + 1) \tag{12}$$

where, in this equation, j is the smallest integer such that u is less than 2π. Then, since the synodic alignment advances by u every synodic period, the set of first synodic alignments that will cause the second synodic alignment to fall in the critical region lags the critical region by u. This set is shown in Fig. 3 and is defined by

$$(v_S | -v_C - u < v_S < v_C - u \text{ or } \pi - v_C - u < v_S < \pi + v_C - u) \tag{13}$$

This region is the "second pass critical region." Extending this approach to multiple passes, the anomaly of the first synodic alignment must be an element of the following set for an intercept to occur in at least one of N passes.

$$\bigcup_{k=1}^{N} C_k = \bigcup_{m=0}^{N-1} (v_S | -v_C - mu < v_S < v_C - mu \text{ or } \pi - v_C - mu < v_S < \pi + v_C - mu) \tag{14}$$

This set and the critical angle are the foundation for the probability calculations and lack only the distribution function for the random variable v_{SI}, the distribution function for the random variable N (the number of synodic alignments), and the conditional relationship of these two random variables with the probability of intercept.

III. Probability Model

The purpose of this section is to define the assumptions and analysis necessary to determine the probability P_I of a given platform intercepting a given target (for at least t_t s) within some specified time T_I of the problem start time, T_0. The problem start time refers to the time when a decision has been made to transmit to a given target. The start time is assumed to be equally likely to occur between the time the platform is placed in orbit and the end of the platform lifetime. This implies that T_0 is distributed uniformly.

Let the synodic alignment that precedes T_0 be called the zeroth synodic alignment (at time t_{S0}), and the synodic alignment that follows T_0 be called the first synodic alignment (at time t_{SI}). The positions of the platform and target at T_0 are v_{P0} and v_{T0}, respectively. Since T_0 is distributed uniformly and the orbits of the target and platform are circular, v_{P0} and v_{T0} are each distributed uniformly between 0 and 2π. The random variable t_{SI} is related to v_{P0} and v_{T0} in the following manner:

$$t_{SI} = \frac{v_{T0} - v_{P0}}{w_P - w_T} \qquad \left(\frac{v_{T0} - v_{P0}}{w_P - w_T} \geq 0\right)$$

or

$$t_{SI} = P_S + \frac{v_{T0} - v_{P0}}{w_P - w_T} \qquad \left(\frac{v_{T0} - v_{P0}}{w_P - w_T} < 0\right) \tag{15}$$

Then t_{SI} is distributed uniformly between zero and P_S. Finally, the angle of the first synodic alignment is given by

$$v_{SI} = w_P t_{SI} + v_{P0} - 2k_P\pi = w_T t_{SI} + v_{T0} - 2k_T\pi \tag{16}$$

Although the sum and difference of two uniformly distributed random variables is triangularly distributed (in this case between 0 and 4π), the parameters k_P and k_T are nonnegative integers chosen so that v_{SI} is between 0 and 2π rather than 0 and 4π. Therefore, v_{SI} also is distributed uniformly between 0 and 2π.

Several other assumptions should be provided before the intercept probability is developed: 1) t_t is small compared to P_S, P_R, and T_I, 2) an intercept is symmetric about the synodic alignment, and 3) for $v_C < 90$ deg, an intercept can be considered as a point event.

Assumption 1 suggests that t_t is a small quantity compared to all other time parameters. Typically this is not a demanding constraint. However, if t_t is near the value of P_R, the approximation for v_C provided earlier will not be valid. Assumption 2 is not true generally. However, the asymmetry is random and typically averages out in Monte Carlo calculations. Assumption 3 generally is true if assumptions 1 and 2 are valid.

The probability of intercept will be calculated for two different cases: 1) when the critical angle is less than 90 deg, and 2) when the critical angle equals 90 deg. In the first case, assumptions 1 and 3 are important, while assumptions 1 and 3 are required for the second case. For $v_C < 90$ deg, the probability of intercept is

$$P_I = \sum_{\text{all } k} P(I/k) P(k) \tag{17}$$

where $P(I/k)$ is the probability of at least one intercept given exactly k passes within T_I, and $P(k)$ is the probability of exactly k passes within T_I. However, for $kP_S < T_I < (k+1)P_S$, $P(k)$ and $P(k+1)$ are the only nonzero probabilities and Eq. (17) reduces to

$$P_I = P(I/k)P(k) + P(I/k+1)P(k+1) \tag{18}$$

All that remains to calculate P_I is to find expressions that describe $P(I/k)$ and $P(k)$ for various choices of k. The approach that is used in the following discussion will consider the special situations where $k=0$ and 1. An attempt then will be made to generalize the resulting expressions.

First, calculate P_I when $k=0$ (i.e., $T_I < P_S$). This requires the calculation of $P(I/0)$, $P(0)$, $P(I/1)$, and $P(1)$. $P(I/0)$ is the probability that an intercept occurs even though a synodic alignment does not occur in the interval $(T_0, T_0 + T_I)$. The only way an intercept can occur given this condition is if the intercept time is large enough (or shifted from the synodic alignment far enough) and T_0 or $T_0 + T_I$ is close enough to a synodic alignment to allow an overlap of the intercept interval and the interval $(T_0, T_0 + T_I)$. For $v_C < 90$ deg, the intercept interval is only t_t long. This supports the application of assumption 3 which, in turn, implies that $P(I/0)$ is zero. Later, when the $v_C = 90$ deg case is considered, $P(I/0)$ will not be a negligible quantity.

$P(I/1)$ can be calculated from Eq. (6) by taking the ratio of the total angular region contained in the set to 2π, or,

$$P(I/1) = \frac{4v_C}{2\pi} = \frac{2v_C}{\pi} \tag{19}$$

Since $T_I < P_S$, the probability of having exactly one intercept is just

$$P(1) = T_I/P_S \tag{20}$$

Then, by Eqs. (18-20),

$$P_I = \frac{2v_C T_I}{\pi P_S} \tag{21}$$

This equation gives the probability of intercept given that T_I is less than P_S and assuming that an intercept is a point event.

The next step is to let $k=1$ (i.e., $P_S < T_I < 2P_S$). This is more easily extended to the situation where $kP_S < T_I < (k+1)P_S$.

For $P_S < T_I < 2P_S$, $P(I/1)$ is unchanged but

$$P(1) = \frac{2P_S - T_I}{P_S} \tag{22}$$

and

$$P(2) = \frac{T_I - P_S}{P_S} \tag{23}$$

The calculation of $P(I/2)$ is somewhat more complicated. The set of first synodic alignments that results in at least one intercept in two passes is given by Eq. (14) with $N=2$. Similar to $P(I/1)$, $P(I/2)$ is determined by the ratio of the angular region in this set to 2π. However, the actual size of the set depends on the amount of overlap in the union of the first- and second-pass critical regions. This, in turn, depends on the synodic phase shift u. Use Fig. 3b to verify the following equations:

$$P(I/2) = \min\left\{1, f_2(v_C, u), \frac{4v_C}{\pi}\right\} \tag{24}$$

where

$$\begin{aligned} f_2(v_C,u) &= g_2(v_C,u) & 0<u\le\pi/2 \\ &= g_2(v_C,\pi-u) & \pi/2<u\le\pi \\ &= g_2(v_C,u-\pi) & \pi<u\le 3\pi/2 \\ &= g_2(v_C,2\pi-u) & 3\pi/2<u\le 2\pi \end{aligned} \tag{25}$$

and

$$g_2(v_C,u) = \frac{2v_C + u}{\pi} \tag{26}$$

The extension of Eqs. (22-25) to a more general case is straightforward but tedious. The expressions for $P(k)$ and $P(k+1)$ extend directly to

$$P(k) = \frac{(k+1)P_S - T_I}{P_S} \tag{27}$$

and

$$P(k+1) = \frac{T_I - kP_S}{P_S} \tag{28}$$

However, the expressions for $P(I/k)$ and $P(I/k+1)$ are not easily generalized. Certainly $P(I/k)$ can be written as

$$P(I/k) = \min\left\{1, f_k(v_C,u), \frac{2kv_C}{\pi}\right\} \tag{29}$$

Unfortunately, the expression for $f_k(v_C,u)$ is difficult to generalize; although, for a given k, it is straightforward to determine. Other examples of $f_k(v_C,u)$ can be defined by finding $g_k(v_C,u)$. For $k=3$,

$$g_3(v_C,u) = \min\left\{\frac{2v_C+2u}{\pi}, \frac{4v_C - 2(u-\pi/2)}{\pi}\right\} \tag{30}$$

and, for $k=4$,

$$g_4(v_C,u) = \min\left\{\frac{2v_C+3u}{\pi}, \frac{4v_C-4(u-\pi/2)}{\pi}, \frac{6v_C-(u-\pi/2)}{\pi}, \frac{6v_C+3u-\pi/3}{\pi}\right\} \tag{31}$$

Although there is some pattern in the comparison of $g_2(v_C,u)$, $g_3(v_c,u)$, and $g_4(v_C,u)$, a generalization is not clear.

Combining Eqs. (18) and (27-29), the probability of a particular platform intercepting a particular target (with $v_C <$ 90 deg) is given in the following expression.

$$P_I = \min\left\{1, f_k(v_C,u), \frac{2kv_C}{\pi}\right\}\left[\frac{(k+1)P_S - T_I}{P_S}\right] + \min\left\{1, f_{k+1}(v_C,u), \frac{2(k+1)v_C}{\pi}\right\}\left[\frac{T_I - kP_S}{P_S}\right] \tag{32}$$

This completes the development of P_I for the case when v_C is less than 90 deg.

When $v_C = 90$ deg any synodic alignment results in an intercept. Therefore, the following discussion will be restricted to the region where there is at most one synodic alignment. For a given synodic alignment, there may be more than one opportunity for an intercept to occur. Figure 4 shows a situation that results in four intercept opportunities for a particular synodic alignment. This suggests that an intercept is not a point event and assumption 3 is no longer valid. If T_0 is

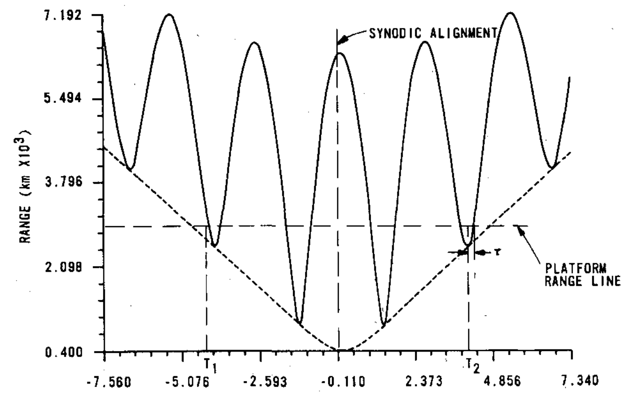

Fig. 4 Intercept opportunities.

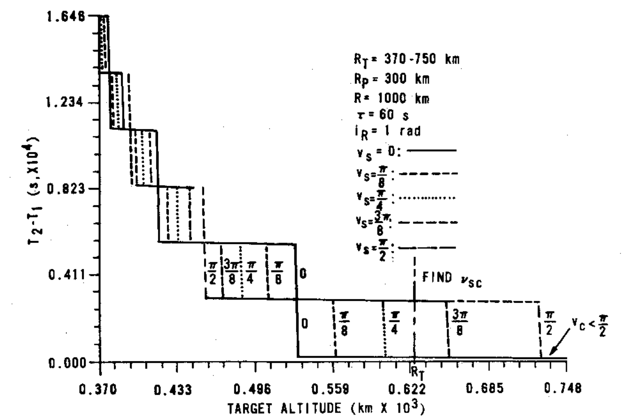

Fig. 5 Intercept interval.

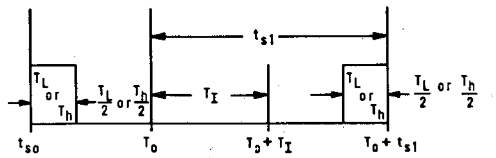

Fig. 6 Start time location.

located anywhere in the interval (T_1, T_2), an intercept will occur, assuming that $T_I > P_R$. The magnitude of $T_2 - T_1$ depends on the orbits of the platform and target, the range of the platform, and v_S. Of particular interest is the variation of $T_2 - T_1$ with v_S. Figure 5 shows that for a given platform-target pair with a given platform range, the value of $T_2 - T_1$ jumps sharply between two values as v_S varies from 0 to 90 deg. Particularly for given R_P, R_T, and R, there is a value of v_S (call it v_{SC}) at which this jump occurs. Assumption 2 now will be invoked. Although the interval $(T_2 - T_1)$ is not generally located symmetrically about the synodic alignment, the amount of asymmetry is usually small compared to P_S. Therefore, assume the symmetry exists and define the lower value of $T_2 - T_1$ to be T_L and the higher value to be T_H. The relationship of T_L and T_H with the variables T_0, t_{SO}, t_{SI}, and T_I is shown in Fig. 6. Finally, this suggests that the probability of intercept is

$$P_I = P(I|T_L,T_H)P(T_L,T_H) + P(I|T_H,T_L)P(T_H,T_L) + P(I|T_L,T_L)P(T_L,T_L) + P(I|T_H,T_H)P(T_H,T_H) \quad (33)$$

where $P(T_L,T_H)$ is the probability that the zeroth synodic alignment has the lower intercept time and the first synodic alignment has the higher intercept time, and $P(I|T_L,T_H)$ is the probability of intercept given that the lower intercept time occurs at the zeroth alignment and the higher time occurs at the first synodic alignment.

Also, from Fig. 5,

$$P(I|T_L,T_L) = \min\left\{\frac{T_I + T_L}{P_S}, 1\right\} \quad (34)$$

$$P(I|T_H,T_H) = \min\left\{\frac{T_I + T_H}{P_S}, 1\right\} \quad (35)$$

$$P(I|T_L,T_H) = \min\left\{\frac{T_I + T_L/2 + T_H/2}{P_S}, 1\right\} \quad (36)$$

and

$$P(T_L,T_H) = P(T_H,T_L) \quad (37)$$

Consider the possibility that $v_{SC} < v_{SO} < 90$. Then,

$$P(T_L) = P(v_{SC} < v_{SO} < 90) \quad (38)$$

and

$$P(T_H) = 1 - P(T_L) \quad (39)$$

where $P(T_L)$ is the probability that the zeroth synodic alignment has an intercept time of T_L, and v_{SO} is the anomaly of the zeroth synodic alignment. If $0 < v_{SO} < v_{SC}$, then Eqs. (38) and (39) are the same except that L and H are interchanged. Since v_{SO} is distributed uniformly,

$$P(T_H) = 2v_{SC}/\pi \quad (40)$$

Substituting Eqs. (34-37) into Eq. (33) and simplifying,

$$P_I = \frac{T_I + T_L P(T_L) + T_H P(T_H)}{P_S} \quad (41)$$

Then, letting $E(T) = T_L P(T_L) + T_H P(T_H)$, Eq. (41) can be written as

$$P_I = \min\left\{1, \frac{T_I + E(T)}{P_S}\right\} \quad (42)$$

The calculation of $E(T)$ requires that v_{SC} be calculated first. Unfortunately, v_{SC} was determined iteratively using a procedure that becomes sensitive as R_T approaches R_P. An alternative is to approximate $E(T)$. This can be done adequately by finding a fit to the family of curves in Fig. 5 for the given range and platform orbit. The probability calculations in this paper used the following expression as an approximate for $E(T)$.

$$E(T) = 2\left\{\frac{1}{w_P - w_T}\cos^{-1}\left[\frac{R_P^2 + R_T^2 - R^2}{R_P R_T(1+\cos i_R)} - \frac{(1-\cos i_R)}{(1+\cos i_R)}\right] - \frac{\pi}{w_P + w_T}\right\} \quad (43)$$

However, this expression for $E(T)$ cannot be applied generally. Other choices of R and R_P may require a different expression for $E(T)$ or an indirect calculation of $E(T)$ using v_{SC}. This completes the development of an expression for P_I when $v_C = 90$ deg.

Figure 7 gives several examples of intercept calculations for a range of 1000 km and various relative inclinations. If this curve were compared to a similar calculation using the $v_C < 90$ deg expression for P_I, the curves for R_T greater than about

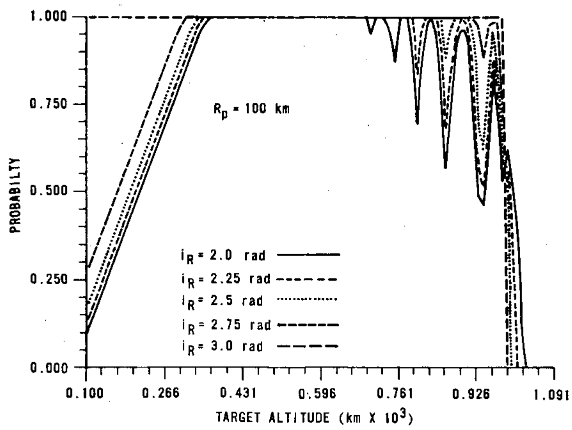

Fig. 7 Large range effects.

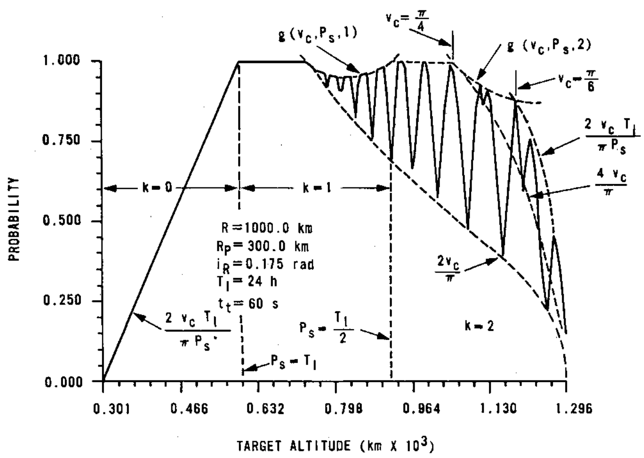

Fig. 8 Single platform intercept probability.

300 km would be unchanged. Also, as the relative inclination decreases, the probability calculations in the linear portion of the curves also approach the $v_C < 90$ deg calculations. This emphasizes an important point. The majority of intercept probability calculations are obtained using the $v_C < 90$ deg expression for P_I. The only time the $v_C = 90$ deg expression is required is when the relative inclination angle is large or the target orbit is close to the platform orbit. In this sense, the $v_C = 90$ deg case can be considered as a "correction" to the $v_C < 90$ deg case. All that remains is to investigate some of the intercept probability calculations, characterize the variations in probability, and compare the results to Monte Carlo calculations. This will be done in the next section.

IV. Calculations and Conclusions

The results of the preceding three sections can be summarized by noting that the intercept probability is given by Eq. (42) whenever there is at most one pass (i.e., $T_I < P_S$), and the intercept probability is given by Eq. (32) whenever there is at least one pass (i.e., $T_I > P_S$). The curves that result from these two equations, such as the ones in Figs. 7 and 8, are rather complex. However, there do seem to be smooth envelopes which bound the oscillating portion of the curve from above and below. This suggests that there may be an easier way of characterizing the probability curves. This characterization will be accomplished in this section by finding expressions that define the upper bound of the curves, the lower bound of the oscillating portion of the curves, and the locus of local minima. It is hoped that this characterization will provide a better understanding of the variations in the intercept probability curves and provide a simpler set of expressions for describing the general shape of the curves.

The rapid oscillations of the curves in Figs. 7 and 8 are a result of the combined effects of the synodic phase shift [Eq. (12)] and the size of the critical region (defined by v_C). From Eqs. (29) and (32) it is clear that the source of oscillations in the probability curves is $f(v_C, k)$. Then the maximum probability of intercept given exactly k passes is

$$P_{\max}(I/k) = \min\left\{1, \frac{2kv_C}{\pi}\right\} \tag{44}$$

When this equation is combined with Eq. (32) there are three possibilities for maximum intercept probability, P_{IMX}. One possibility is that $2kv_C/\pi$ is greater than 1. In this case, $P_{I\max}$ is equal to 1. Another possibility is that $2(k+1)v_C/\pi$ is less than 1. Then $2kv_C/\pi$ is also less than 1 and

$$P_{IMX} = \min\left\{1, \frac{2v_C}{\pi}\left(\frac{T_I}{P_S}\right)\right\} \tag{45}$$

The last possibility is that $2(k+1)v_C/\pi$ is greater than 1 and $2kv_C/\pi$ is less than 1. In this final case,

$$P_{IMX} = \frac{2kv_C}{\pi}\left[\frac{(k+1)P_S - T_I}{P_S}\right] + \frac{T_I - kP_S}{P_S} \tag{46}$$

Let $g(v_C, P_S, T_I)$ be the expression on the right-hand side of Eq. (46). Then the maximum intercept probability can be described by

$$P_{IMX} = \min\left\{1, \frac{2v_C T_I}{\pi P_S}, g(v_C, P_S, k)\right\} \tag{47}$$

This expression is shown as the upper envelope of the intercept probability curve in Fig. 8. The lower envelope corresponds to the minimum probability of intercept and can be easily determined as

$$P_{IMN} = \frac{2v_C}{\pi}\min\left\{1, \frac{T_I}{P_S}\right\} \tag{48}$$

The intercept probability reaches this minimum during the oscillating part of the curve whenever the synodic phase shift equals π or 2π. This corresponds to all of the critical regions (second pass, third pass, etc.) exactly overlapping the first-pass critical region. Figure 3 may be helpful in visualizing this. It is this type of critical region overlap that causes all local minima. In general, all local minima will be found along the curves

$$h(k, v_C) = 2kv_C/\pi \tag{49}$$

as long as $2kv_C/\pi$ is less than P_{IMX}. Examples of these curves are shown in Figs. 8 and 9.

The primary result of the preceding discussion is that the probability of intercept curve can be characterized by a few simple expressions: Eqs. (47-49). These expressions do not describe the oscillations in probability, but they do provide useful upper and lower bounds. Furthermore, the minima of the oscillations can be located precisely by merely monitoring values of u and determining whether $h(k, v_C)$ is less than P_{IMX}. Therefore, the shape and basic variation of P_I as a function of R_T can be described almost completely except for the precise location of the maxima. This completes the characterization of the probability curves.

The next step is to compare the predictions of intercept probability to estimates obtained with Monte Carlo runs. The Monte Carlo runs sampled random initial anomaly angles for the target and platform from a uniform distribution. The trajectories of the target and platform were then followed for a

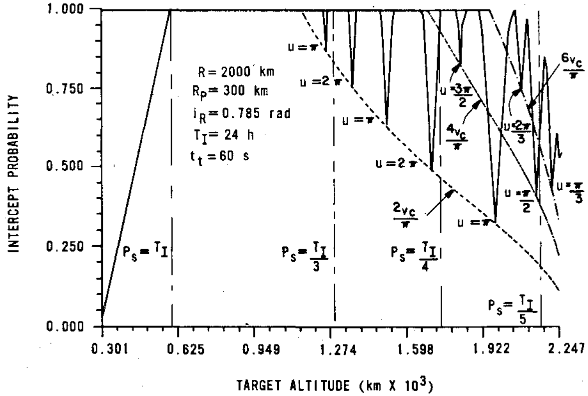

Fig. 9 Oscillation minima.

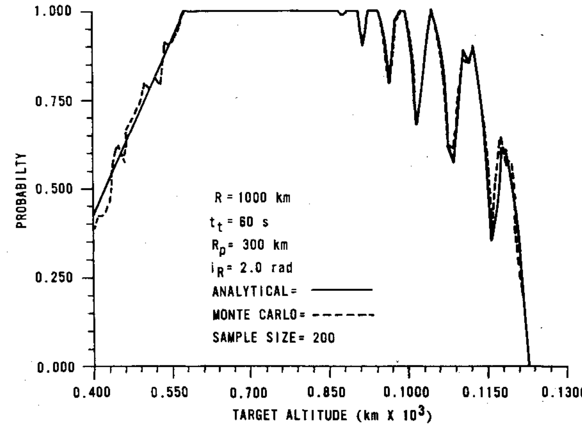

Fig. 10 Monte Carlo comparison with corrected calculation.

time T_I or until an intercept was observed. Results of these runs are given in Fig. 10 along with the predicted values. The 200 samples noted in the figure refer to 200 separate trajectories being run. The number of trajectories that result in intercepts is then divided by 200 to obtain an estimate of P_I. The comparison is quite close.

V. Conclusions

This paper presents an approach to calculating the intercept probabilities of satellites in circular orbits that compares favorably to Monte Carlo estimates. The expressions obtained provide a useful way of investigating the variation of intercept probability with any of the orbital and intercept parameters; some of which were provided in this paper. In general, a completely analytical expression for intercept probability is not available, and a combined analytical/computational approach is required. However, within the limitations of the problem, algebraic expressions are provided that replace numerical estimates with acceptable results. An interesting class of problems that could be addressed using the results in this paper are various optimal placement problems that require that a particular constellation of platform satellites be able to interact (within a given period of time and for a given period of time) with a random placement of target satellites located in some specified region of space. Future studies of intercept probability should consider effects of perturbations and elliptical orbits on intercept probability calculations.

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